

# Wide Gaps with Short Extenders

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## Abstract

The paper is a continuation of [Gi]. Extending the methods of [Gi] we show the following: Let  $\kappa = \bigcup_{n < \omega} \kappa_n$ ,  $\kappa_0 < \kappa_1 < \dots < \kappa_n < \dots$

(1) if each  $\kappa_n$  carries an extender of the length of the first Mahlo above  $\kappa_n$ , *then* for every  $\lambda > \kappa$  there is a generic extension satisfying  $2^\kappa \geq \lambda$ ;

(2) if each  $\kappa_n$  carries an extender of the length of the first fixed point of the  $\aleph$ -function above  $\kappa_n$  of order  $n$ , *then* for every  $\lambda$ ,  $\kappa < \lambda < \text{least inaccessible above } \kappa$  there is a generic extension satisfying  $2^\kappa \geq \lambda$ .

## 0 Introduction

We would like to extend the method of [Gi] to deal with arbitrary gaps between a singular cardinal  $\kappa$  and its power. The main structure of [Gi] was as follows:

Let  $\kappa = \bigcup_{n < \omega} \kappa_n$ ,  $\kappa_0 < \kappa_1 < \dots < \kappa_n < \dots$  and we want to make  $2^\kappa \geq \kappa^{+\delta+1}$  for some  $\delta < \kappa_0$ .

- (1) For every  $\nu \leq \delta$  we use  $\langle \kappa_n^{+n+\nu+1} \mid n < \omega \rangle$  as a sequence below  $\kappa$  responsible for  $\kappa^{+\nu+1}$ .
- (2) Sequences of elementary submodels of some  $H(\chi)$  ( $\chi$  big enough)  $\langle A^{0\tau} \mid \tau \leq \delta \rangle$  with  $A^{0\tau} \subseteq A^{0\tau'}$  ( $\tau \leq \tau' \leq \delta$ ) and  $|A^{0\tau}| = \kappa^{+\tau+1}$  were used to gradually shrink the number of possible choices of generic  $\omega$ -sequences. Actually, the first submodels  $A^{00}$ 's are the most important, since they have the size  $\kappa^+$  which insures eventually the  $\kappa^{++}$ -c.c. of the final forcing. It was essential that  $A^{0\tau}$ 's include  $\delta$  and we can deal at once with  $\langle \sup(A^{0\tau} \cap \kappa^{+\nu+1}) \mid \nu \leq \delta \rangle$ .

Stretching the forcing of [Gi] slightly, we can deal with  $\delta$ 's above  $\kappa$  but below  $\kappa^{++}$  and to make  $2^\kappa$  as big as we like below  $\kappa^{+(\kappa^{++})} = \aleph_{\kappa+\kappa^{++}} = \aleph_{\kappa^{++}}$ . Thus for example if  $\delta = \kappa^+ + 1$ , then in (1) we may use  $\langle \kappa_n^{+\kappa_n^++\kappa_n^++1} \mid n < \omega \rangle$  instead of  $\langle \kappa_n^{+n+\delta+1} \mid n < \omega \rangle$  and take care of

$\kappa^{+\kappa^{++}+1}$ . This will leave enough room for  $\nu$ 's below  $\delta$  as well. There is no problem with (2), since the number of cardinals we are dealing with is still below  $\kappa^{++}$ . In Section 2 we will do it under weaker assumptions.

But once we are above  $\kappa^{++}$ , (1) and especially (2), require essential changes.

We show in the first section the following:

**Theorem 1\*** *Let  $\kappa = \bigcup_{n<\omega} \kappa_n$ ,  $\kappa_0 < \kappa_1 < \dots < \kappa_n < \dots$  and each  $\kappa_n$  carries an extender of the length of the first Mahlo above  $\kappa_n$ . Then for every  $\lambda > \kappa$  there is a cofinality preserving extension, with the exception of the successors of inaccessibles above  $\kappa$ , not adding new bounded subsets to  $\kappa$  and satisfying  $2^\kappa \geq \lambda$ .*

In the third section we show the following:

**Theorem 2** *Let  $\kappa = \bigcup_{n<\omega} \kappa_n$ ,  $\kappa_0 < \kappa_1 < \dots < \kappa_n < \dots$  and each  $\kappa_n$  carries an extender of the length of the first fixed point of the  $\aleph$ -function above  $\kappa_n$  of order  $n$ . Then for every  $\lambda > \kappa$  and below the first inaccessible above  $\kappa$  there is a cofinality preserving extension not adding new bounded subsets to  $\kappa$  and satisfying  $2^\kappa \geq \lambda$ .*

## 1 Getting an arbitrary gap

Our purpose here will be to define the forcing to prove Theorem 1. We will use forcings similar to those of [Gi, Sec. 4] with changes for overcoming (1) and (2) above.

Fix an ordinal  $\delta > 1$ .

**Definition 1.1** The set  $\mathcal{P}'$  consists of pairs  $\langle \langle A^{0\tau}, A^{1\tau} \rangle \mid \tau \leq \delta \rangle$  so that the following holds:

(1) for every  $\tau \leq \delta$   $A^{0\tau}$  is an elementary submodel of  $\langle H(\kappa^{+\delta+2}), \epsilon, \langle \kappa^{+i} \mid i \leq \delta + 2 \rangle \rangle$  such that

(a)  $|A^{0\tau}| = \kappa^{+\tau+1}$  and  $A^{0\tau} \supseteq \kappa^{+\tau+1}$  unless for some  $n < \omega$  and an inaccessible  $\tau'$ ,  $\tau = \tau' + n$  and then  $|A^{0\tau}| = \kappa^{+\tau}$  and  $A^{0\tau} \supseteq \kappa^{+\tau}$

(b)  $|A^{0\tau}| > A^{0\tau} \subseteq A^{0\tau}$

(2) for every  $\tau < \tau' \leq \delta$ ,  $A^{0\tau} \subseteq A^{0\tau'}$

(3) for every  $\tau \leq \delta$ ,  $A^{1\tau}$  is a set of at most  $\kappa^{+\tau+1}$  elementary submodels of  $A^{0\tau}$  so that

(a)  $A^{0\tau} \in A^{1\tau}$

(b) if  $B, C \in A^{1\tau}$  and  $B \subsetneq C$  then  $B \in C$

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\*S. Shelah suggested to push this down combining the forcing of the theorem with [Gi-Ma, Sec. 2] or [Gi1, Sec. 4]. It is possible this way to obtain models with no Mahlo below  $\kappa$  and  $2^\kappa$  arbitrary large.

- (c) if  $B \in A^{1\tau}$  is a successor point of  $A^{1\tau}$  then  $B$  has at most two immediate predecessors under the inclusion and is closed under  $\kappa^{+\tau}$ -sequences.
- (d) let  $B \in A^{1\tau}$  then either  $B$  is a successor point of  $A^{1\tau}$  or  $B$  is a limit element and then there is a closed chain of elements of  $B \cap A^{1\tau}$  unbounded in  $B \cap A^{1\tau}$  and with limit  $B$ .
- (e) for every  $\tau', \tau \leq \tau' \leq \delta$ ,  $A \in A^{1\tau}$  and  $B \in A^{1\tau'}$  either  $B \supseteq A$  or there are  $\ell < \omega$  and  $\tau'_1, \tau'_2, \dots, \tau'_\ell$ ,  $\tau \leq \tau'_1 \leq \dots \leq \tau'_\ell \leq \delta$ ,  $B_1 \in A \cap A^{1\tau'_1}, \dots, B_\ell \in A \cap A^{1\tau'_\ell}$  such that

$$B \cap A = B_1 \cap \dots \cap B_\ell \cap A ,$$

if  $\tau = \tau'$ , then we can pick  $\tau'_1$  (and hence all the rest) above  $\tau$ .

- (f) let  $A$  be an elementary submodel of  $H(\kappa^{+\delta+2})$  of cardinality  $|A^{0\tau}|$ , closed under  $< |A^{0\tau}|$ -sequences,  $|A^{0\tau}| \in A$  and including  $\langle \langle A^{0\tau'}, A^{1\tau'} \rangle \mid \tau' \leq \delta \rangle$  as an element, for some  $\tau \leq \delta$ . Then for every  $\tau', \tau \leq \tau' \leq \delta$  and  $B \in A^{1\tau'}$  either  $B \supseteq A$  or there are  $\tau'_1, \dots, \tau'_\ell$ ,  $\tau \leq \tau'_1 \leq \dots \leq \tau'_\ell \leq \delta$ ,  $B_1 \in A \cap A^{1\tau'_1}, \dots, B_\ell \in A \cap A^{1\tau'_\ell}$  such that

$$B \cap A = B_1 \cap \dots \cap B_\ell \cap A .$$

The present definition of  $\mathcal{P}'$  differs from those of [Gi, 4.1] by two additional conditions (f) and (g). They are desired in order to overcome the difficulty (2). This way the number of possible intersections (and actually intersections themselves) is controlled.

The addition of (f) and (g) makes the proof of distributivity of the forcing more involved and we shall further concentrate on this matter.

The definition of order 1.2, 1.3 and 1.4 repeats the corresponding ones in [Gi, Sec. 4].

Let for  $\tau \leq \delta$   $A_{in}^{1\tau}$  be the set  $\{B \cap B_1 \cap \dots \cap B_n \mid B \in A^{1\tau}, n < \omega \text{ and } B_i \in A^{1\rho_i} \text{ for some } \rho_i, \tau < \rho_i \leq \delta \text{ for every } i, 1 \leq i \leq n\}$ .

**Definition 1.2** Let  $\langle \langle A^{0\tau}, A^{1\tau} \rangle \mid \tau \leq \delta \rangle$  and  $\langle \langle B^{0\tau}, B^{1\tau} \rangle \mid \tau \leq \delta \rangle$  be elements of  $\mathcal{P}'$ . Then  $\langle \langle A^{0\tau}, A^{1\tau} \rangle \mid \tau \leq \delta \rangle \geq \langle \langle B^{0\tau}, B^{1\tau} \rangle \mid \tau \leq \delta \rangle$  iff for every  $\tau \leq \delta$

- (1)  $A^{1\tau} \supseteq B^{1\tau}$
- (2) for every  $A \in A^{1\tau}$  either
  - (a)  $A \supseteq B^{0\tau}$  or
  - (b)  $A \subset B^{0\tau}$  and then  $A \in B^{1\tau}$  or
  - (c)  $A \not\supseteq B^{0\tau}$ ,  $B^{0\tau} \not\supseteq A$  and then  $A \in B_{in}^{1\tau}$ .

**Definition 1.3** Let  $\tau \leq \delta$ . Set  $\mathcal{P}'_{\geq \tau} = \{\langle \langle A^{0\rho}, A^{1\rho} \rangle \mid \tau \leq \rho \leq \delta \rangle \mid \exists \langle \langle A^{0\nu}, A^{1\nu} \rangle \mid \nu < \tau \rangle \langle \langle A^{0\nu}, A^{1\nu} \rangle \mid \nu < \tau \rangle \frown \langle \langle A^{0\rho}, A^{1\rho} \rangle \mid \tau \leq \rho \leq \delta \rangle \in \mathcal{P}'\}$ .

Let  $G(\mathcal{P}'_{\geq \tau}) \subseteq \mathcal{P}'_{\geq \tau}$  be generic. Define  $\mathcal{P}'_{< \tau} = \{\langle \langle A^{0\nu}, A^{1\nu} \rangle \mid \nu < \tau \rangle \mid \exists \langle \langle A^{0\rho}, A^{1\rho} \rangle \mid \tau \leq \rho \leq \delta \rangle \in G(\mathcal{P}'_{\geq \tau}) \langle \langle A^{0\nu}, A^{1\nu} \rangle \mid \nu < \tau \rangle \frown \langle \langle A^{0\rho}, A^{1\rho} \rangle \mid \tau \leq \rho \leq \delta \rangle \in \mathcal{P}'\}$ .

The following lemma is obvious

**Lemma 1.4**  $\mathcal{P}' \simeq \mathcal{P}'_{\geq \tau} * \mathcal{P}'_{< \tau} \quad (\tau \leq \delta)$ .

Let us now define the main preparation forcing  $\mathcal{P}$ . The definition repeats mainly the corresponding definition of [Gi], 4.14. The only changes are designed to overcome (1), since we are now probably dealing with a large number of cardinals above  $\kappa$ .

Thus we like cardinals to correspond to cardinals, regular cardinals to regular cardinals, and limit cardinals to correspond to limit cardinals. This puts some limitations. In particular, inaccessibles should correspond to inaccessibles, inaccessibles of order 1 (i.e. inaccessibles which are limits of inaccessibles) should correspond to inaccessibles of order  $\geq 1$ , etc. We shall arrange in a moment sets of inaccessibles below  $\kappa$  corresponding to those above  $\kappa$ . But first we destroy all Mahlo cardinals above  $\kappa$  by forcing for each such cardinal a club avoiding inaccessibles.

Fix  $n < \omega$ . We now want to define “good” inaccessibles. This notion will be similar to the notion of good ordinals used in [Gi, 2.8]. Let  $\chi_n$  denote the least Mahlo cardinal above  $\kappa_n$ . For every  $k \leq n$  we consider the structure  $\mathfrak{a}_{n,k} = \langle H(\chi_n^{+k+2}), \epsilon, E_n, 0, 1, \dots, \alpha, \dots \mid \alpha < \kappa_n^{+k+2} \rangle$ . For an ordinal  $\xi < \chi_n$  let  $tp_{n,k}(\xi)$  be the type realized by  $\xi$  in  $\mathfrak{a}_{n,k}$ .

The following lemma is obvious:

**Lemma 1.4.1** *There are stationary many below  $\chi_n$  inaccessible cardinals  $\xi$  so that*

- (1) *the set  $\{\delta < \xi \mid tp_{n,n}(\delta) = tp_{n,n}(\xi)\}$  is unbounded in  $\xi$ ,*
- (2)  *$\xi = A \cap \chi_n$  for some  $A \prec \mathfrak{a}_{n,n}$ .*

We fix an inaccessible  $\xi_n$  satisfying the conclusion of the lemma.

**Definition 1.4.2** An inaccessible cardinal  $\delta < \xi_n$  is called  $k$ -good (for some  $k \leq n$ ) if  $tp_{n,k}(\delta) = tp_{n,k}(\xi_n)$ .

The next lemma is obvious.

**Lemma 1.4.3** *If  $\delta$  is  $k$ -good for some  $k > 1$ , then  $\delta$  is the limit of  $k-1$ -good inaccessibles.*

Further we shall use only the restriction of the extender  $E_n$  to  $\xi_n$ . Also each inaccessible  $\lambda > \kappa$  will correspond to a sequence  $\langle \delta_n \mid n < \omega \rangle$ , where every  $\delta_n$  is  $k_n$ -good inaccessible and  $k_0 < k_1 < \dots < k_n < \dots$ .

**Definition 1.5** The set  $\mathcal{P}$  consists of sequences of triples  $\langle \langle A^{0\tau}, A^{1\tau}, F^\tau \rangle \mid \tau \leq \delta \rangle$  so that the following holds:

- (0)  $\langle \langle A^{0\tau}, A^{1\tau} \rangle \mid \tau \leq \delta \rangle \in \mathcal{P}'$
- (1) for every  $\tau_1 \leq \tau_2 \leq \delta$ ,  $F^{\tau_1} \subseteq F^{\tau_2} \subseteq \mathcal{P}^*$
- (2) for every  $\tau \leq \delta$ ,  $F^\tau$  is as follows:
  - (a)  $|F^\tau| = |A^{0\tau}|$
  - (b) for every  $p = \langle p_n \mid n < \omega \rangle \in F^\tau$  if  $n < \ell(p)$  then every  $\alpha$  appearing in  $p_n$  is in  $A^{0\tau} \cup \{|A^{0\tau}|\}$ ; if  $n \geq \ell(p)$  and  $p_n = \langle a_n, A_n, f_n \rangle$  then every  $\alpha$  appearing in  $f_n$  is in  $A^{0\tau} \cup \{|A^{0\tau}|\}$  and
    - (i)  $\text{dom} a_n \cap On \subseteq (A^{0\tau} \cap \kappa^{+\delta+2}) \cup \{|A^{0\tau}|\}$
    - (ii)  $\text{dom} a_n \setminus On$  consists of elements of the following sets:  $\{B \subseteq A^{0\tau} \mid \kappa^+ \leq |B| < |A^{0\tau}|\}$ ,  $A^{1\tau}$  and  $A_{in}^{1\tau}$  such that the elements of the last two sets are closed under  $> |A^{0\tau}|$ -sequences of its elements. If  $\tau = 0$ , then the first set is empty.
  - (c) the largest (under inclusion) element of  $\text{dom} a_n \setminus On$  belongs to  $A^{1\tau}$  and every element of  $\text{dom} a_n$  belongs to it.

Let us further denote this element as  $\max^1(p_n)$  or  $\max^1(a_n)$ .

  - (d) if  $B \in \text{dom} a_n \setminus On$ , then  $a_n(B)$  is an elementary submodel of  $a_{n,k_n}$  of Section 2 of [Gi] with  $3 \leq k_n \leq n$ , including also  $\delta$  as a constant. We require that
    - (d1) if  $|B|$  is a successor cardinal, then  $|a_n(B)| = \kappa_n^{+\tau'+1}$  and  $\kappa_n^{+\tau'}(a_n(B)) \subseteq a_n(B)$ , where  $\tau'$ ,  $\kappa_n^+ < \tau' < \xi_n$  is  $k_n$ -good cardinal,  $k_n$ 's are increasing with  $n$  and  $a_n(|B|) = \kappa_n^{+\tau'+1}$ .
    - (d2) if  $|B|$  is an inaccessible cardinal, then  $|a_n(B)|$  is a  $k_n$ -good inaccessible with  $k_n$ 's increasing with  $n$ ,  $|a_n(B)| > a_n(B) \subseteq a_n(B)$  and  $a_n(|B|) = |a_n(B)|$ .
  - (e) if  $B \in \text{dom} a_n \setminus On$  and  $\alpha \in \text{dom} a_n \cap A^{0\tau}$  then  $a_n(\alpha) \in a_n(B)$  iff  $\alpha \in B$
  - (f) if  $B, C \in \text{dom} a_n \setminus On$  then
    - (f1)  $B \in C$  iff  $a_n(B) \in a_n(C)$
    - (f2)  $B \subset C$  iff  $a_n(B) \subset a_n(C)$ .

The next condition deals with cofinalities correspondence

  - (g) (i) if  $\alpha \in \text{dom} a_n$  and  $cf \alpha \leq \kappa^+$  then  $cf a_n(\alpha) \leq \kappa_n^{+n+1}$ .

- (ii) if  $\alpha \in \text{dom}a_n$  and  $cf\alpha = \kappa^{+\rho}$  for some  $\rho$ ,  $1 \leq \rho \leq \delta + 1$  then  $\kappa^{+\rho} \in \text{dom}a_n$ ,  $cf a_n(\alpha) = a_n(\kappa^{+\rho})$  and for every  $B \in \text{dom}a_n \setminus On$  of cardinality  $\kappa^{+\rho}$   $|a_n(B)| = a_n(\kappa^{+\rho})$ .
- (iii) if  $\alpha \in \text{dom}a_n$  is an inaccessible, then  $a_n(\alpha)$  is  $k_n$ -good inaccessible, with  $k_n$ 's increasing with  $n$ .
- (h) if  $p \in F^\tau$  and  $q \in \mathcal{P}^*$  is equivalent to  $p$  ( $q \leftrightarrow p$ ) with witnessing sequence  $\langle k_n \mid n < \omega \rangle$  starting with  $k_0 \geq 4$  then  $q \in F^\tau$ .
- (i) if  $p = \langle p_n \mid n < \omega \rangle \in F^\tau$  and  $q = \langle q_n \mid n < \omega \rangle \in \mathcal{P}^*$  are such that
  - (i)  $\ell(p) = \ell(q)$
  - (ii) for every  $n < \ell(p)$   $p_n = q_n$
  - (iii) for every  $n \geq \ell(p)$   $a_n = b_n$  and  $\text{dom}g_n \subseteq A^{0\tau}$  where  $p_n = \langle a_n, A_n, f_n \rangle$ ,  $q_n = \langle b_n, B_n, g_n \rangle$
 then  $q \in F^\tau$ .
- (k) if  $p = \langle p_n \mid n < \omega \rangle \in F^\tau$   $q = \langle q_n \mid n < \omega \rangle \in \mathcal{P}^*$  are such that
  - (i)  $\ell(q) \geq \ell(p)$
  - (ii) for every  $n \geq \ell(q)$   $p_n = q_n$
  - (iii) every  $\alpha$  appearing in  $q_n$  for  $n < \ell(q)$  is in  $A^{0\tau}$
 then  $q \in F^\tau$ .

The meaning of the last two conditions is that we are free to change inside  $A^{0\tau}$  all the components of  $p$  except  $a_n$ 's.

- (l) for every  $q \in F^\tau$  and  $\alpha \in A^{0\tau}$  there is  $p \in F^\tau$   $p = \langle p_n \mid n < \omega \rangle$ ,  $p_n = \langle a_n, A_n, f_n \rangle$  ( $n \geq \ell(p)$ ) such that  $p \geq^* q$  and  $\alpha \in \text{dom}a_n$  starting with some  $n_0 < \omega$ .
- (m) for every  $q \in F^\tau$  and  $B \in A^{1\tau} \cup A_{in}^{1\tau}$  as in (b)(ii), there is  $p \in F^\tau$   $p = \langle p_n \mid n < \omega \rangle$ ,  $p_n = \langle a_n, A_n, f_n \rangle$  ( $n \geq \ell(p)$ ) such that  $p \geq^* q$  and  $B \in \text{dom}a_n$  starting with some  $n_0 < \omega$ . Also, this  $p$  is obtained from  $q$  by adding only  $B$  and the ordinals needed to be added after adding  $B$ .
- (n) Let  $p, q \in F^\tau$  be so that
  - (i)  $\ell(p) = \ell(q)$
  - (ii)  $\max^1(p_n) = \max^1(q_n)$ ,  $\max^1(q_n) = \max^1(q_m)$  and  $\max^1(q_n) \in \text{dom}a_n$ , where  $n, m \geq \ell(p)$ ,  $p_n = \langle a_n, A_n, f_n \rangle$ ,  $q_n = \langle b_n, B_n, g_n \rangle$
  - (iii)  $p_n = q_n$  for every  $n < \ell(p)$
  - (iv)  $f_n, g_n$  are compatible for every  $n \geq \ell(p)$

(v)  $a_n \restriction \max^1(q_n) \subseteq b_n$  for every  $n \geq \ell(p)$ , where

$$a_n \restriction B = \{\langle t \cap B, s \cap a_n(B) \rangle \mid \langle t, s \rangle \in a_n\}$$

then the union of  $p$  and  $q$  is in  $F^\tau$  where the union is defined in obvious fashion taking  $p_n \cup q_n$  for  $n < \ell(p)$ , we take at each  $n \geq \ell(p)$   $a_n \cup b_n$ ,  $f_n \cup g_n$  etc.

(o) there is  $F^{\tau*} \subseteq F^\tau$  dense in  $F^\tau$  under  $\leq^*$  such that every  $\leq^*$ -increasing sequence of elements of  $F^{\tau*}$  having the union in  $\mathcal{P}^*$  has it also in  $F^\tau$ . We require that  $F^{\tau*}$  will be closed under the equivalence relation  $\leftrightarrow$ .

(p) let  $p = \langle p_n \mid n < \omega \rangle \in F^\tau$  and  $p_n = \langle a_n, A_n, f_n \rangle$  ( $\ell(p) \leq n < \omega$ ). If for every  $n$ ,  $\omega > n \geq \ell(p)$   $B \in \text{dom} a_n \setminus \text{On}$ ,  $|B| = \kappa^{+\tau+1}$  or  $B \in A^{1\tau'}$  for some  $\tau' \leq \tau$ , then  $p \restriction B \in F^{\tau'}$ , where  $p \restriction B = \langle p_n \restriction B \mid n < \omega \rangle$  and for every  $n < \ell(p)$   $p_n \restriction B$  is the usual restriction of the function  $p_n$  to  $B$ ; if  $n \geq \ell(p)$  then  $p_n \restriction B = \langle a_n \restriction B, B_n, f_n \restriction B \rangle$  with  $a_n \restriction B$  defined in (n)(v),  $f_n \restriction B$  is the usual restriction and  $B_n$  is the projection of  $A_n$  by  $\pi_{\max p_n, B}$ .

(q) let  $p = \langle p_n \mid n < \omega \rangle \in F^\tau$ ,  $p_n = \langle a_n, A_n, f_n \rangle$  and  $A^{0\tau} \notin \text{dom} a_n$  ( $\omega > n \geq \ell(p)$ ). Let  $\langle \sigma_n \mid \omega > n \geq \ell(p) \rangle$  be so that

(i)  $\sigma_n \prec \mathfrak{a}_{n, k_n}$  and  $|\sigma_n|$  is  $k_n$ -good for every  $n \geq \ell(p)$

(ii)  $\langle k_n \mid n \geq \ell(p) \rangle$  is increasing

(iii)  $k_0 \geq 5$

(iv)  $|\sigma_n| > \sigma_n \subseteq \sigma_n$  for every  $n \geq \ell(p)$

(v)  $\text{rng} a_n \in \sigma_n$  for every  $n \geq \ell(p)$ .

Then the condition obtained from  $p$  by adding  $\langle A^{0\tau}, \sigma_n \rangle$  to each  $p_n$  with  $n \geq \ell(p)$  belongs to  $F^\tau$ .

(r) if  $A$  is an elementary submodel of  $H(\kappa^{+\delta+2})$  of a regular cardinality  $\kappa^{+\rho}$ , closed under  $< \kappa^{+\rho}$ -sequences and including  $\langle \langle A^{0\tau'}, A^{1\tau'} \rangle \mid \tau' \leq \delta \rangle$  for some  $\rho < \tau$ , then  $A$  is addable to any  $p \in F^\tau \cap A$ , with the maximal element of  $\text{dom} a_n$ 's  $A^{0\tau}$ , i.e.  $A \cap A^{0\tau}$  can be added to  $p$  remaining in  $F^\tau$ .

**Definition 1.6** Let  $\langle \langle A^{0\tau}, A^{1\tau}, F^\tau \rangle \mid \tau \leq \delta \rangle$  and  $\langle \langle B^{0\tau}, B^{1\tau}, G^\tau \rangle \mid \tau \leq \delta \rangle$  be in  $\mathcal{P}$ . We define

$$\langle \langle A^{0\tau}, A^{1\tau}, F^\tau \rangle \mid \tau \leq \delta \rangle > \langle \langle B^{0\tau}, B^{1\tau}, G^\tau \rangle \mid \tau \leq \delta \rangle$$

iff

(1)  $\langle\langle A^{0\tau}, A^{1\tau} \rangle \mid \tau \leq \delta \rangle > \langle\langle B^{0\tau}, B^{1\tau} \rangle \mid \tau \leq \delta \rangle$  in  $\mathcal{P}'$

(2) for every  $\tau \leq \delta$

(a)  $F^\tau \supseteq G^\tau$

(b) for every  $p \in F^\tau$  and  $B \in B^{1\tau} \cup B_{in}^{1\tau}$  if for every  $n \geq \ell(p)$   $B \in \text{dom} a_n$  then  $p \restriction B \in G^\tau$ , where the restriction is as defined in 1.5(p),  $p = \langle p_n \mid n < \omega \rangle$ ,  $p_n = \langle a_n, A_n, f_n \rangle$  for  $n \geq \ell(p)$ .

**Definition 1.7** Let  $\tau \leq \delta$ . Set  $\mathcal{P}_{\geq \tau} = \{ \langle\langle A^{0\rho}, A^{1\rho}, F^\rho \rangle \mid \tau \leq \rho \leq \delta \rangle \mid \exists \langle\langle A^{0\nu}, A^{1\nu}, F^\nu \rangle \mid \nu < \tau \rangle \langle\langle A^{0\nu}, A^{1\nu}, F^\nu \rangle \mid \nu < \tau \rangle \cap \langle\langle A^{0\rho}, A^{1\rho}, F^\rho \rangle \mid \tau \leq \rho \leq \delta \rangle \in \mathcal{P} \}$ .

Let  $G(\mathcal{P}_{\geq \tau}) \subseteq \mathcal{P}_{\geq \tau}$  be generic. Define  $\mathcal{P}_{< \tau} = \{ \langle\langle A^{0\nu}, A^{1\nu}, F^\nu \rangle \mid \nu < \tau \rangle \mid \exists \langle\langle A^{0\rho}, A^{1\rho}, F^\rho \rangle \mid \tau \leq \rho \leq \delta \rangle \in G(\mathcal{P}_{\geq \tau}) \langle\langle A^{0\nu}, A^{1\nu}, F^\nu \rangle \mid \nu < \tau \rangle \cap \langle\langle A^{0\rho}, A^{1\rho}, F^\rho \rangle \mid \tau \leq \rho \leq \delta \rangle \in \mathcal{P} \}$ .

The following lemma is obvious

**Lemma 1.8**  $\mathcal{P} \simeq \mathcal{P}_{\geq \tau} * \mathcal{P}_{< \tau}$  for every  $\tau \leq \delta$ .

**Lemma 1.9** For every  $\tau \leq \delta$ ,  $\mathcal{P}_{\geq \tau}$  is  $\kappa^{+\tau+2}$ -strategically closed. Moreover, if there are an inaccessible  $\tau' \leq \tau$  and  $n < \omega$  such that  $\tau = \tau' + n$ , then  $\mathcal{P}_{\geq \tau}$  is  $\kappa^{+\tau+1}$ -strategically closed.

**Proof.** The proof is similar to [Gi, 4.18] and we concentrate on new points caused by the additions to the definition of  $\mathcal{P}'$  here, i.e. (f) and (g) of 1.1.

Fix  $\tau \leq \delta$ . Let  $\langle\langle A_i^{0\rho}, A_i^{1\rho}, F_i^\rho \rangle \mid i < i^* \rangle$  be an increasing sequence of conditions in  $\mathcal{P}_{\geq \tau}$  already generated by playing the game. We need to define the move  $\langle\langle A_{i^*}^{0\rho}, A_{i^*}^{1\rho}, F_{i^*}^\rho \rangle \mid \tau \leq \rho \leq \delta \rangle$  of Player I at stage  $i^*$ .

Define this triple by induction on  $\rho$ . The definition of  $F_{i^*}^\rho$  completely repeats the one in [Gi, 4.18]. So we deal only with  $\langle A_{i^*}^{0\rho}, A_{i^*}^{1\rho} \rangle$ .

**Case 1**  $i^*$  is a limit and  $\text{cf } i^* = \kappa^{+\tau+1}$  (or  $\kappa^{+\tau}$ , if  $\tau = \tau' + n$  for inaccessible  $\tau' \leq \tau$  and  $n < \omega$ ).

We set  $A_{i^*}^{0\tau} = \bigcup_{i < i^*} A_i^{0\tau}$  and  $A_{i^*}^{1\tau} = \bigcup_{i < i^*} A_i^{1\tau} \cup \{A_{i^*}^{0\tau}\}$ . Let  $\rho \in (\tau, \delta]$ . Set  $\tilde{A}_{i^*}^{0\rho}$  to be the closure under the Skolem functions and  $\kappa^{+\rho}$ -sequences (or  $< \kappa^{+\rho}$ -sequences, if  $\rho = \rho^* + n$  for an inaccessible  $\rho^*$  and  $n < \omega$ ) of  $\langle\langle A_i^{j\rho'} \mid i < i^* \rangle \mid \tau \leq \rho' \leq \delta \rangle$  ( $j \in 2$ ) and  $\langle A_{i^*}^{1\rho'} \mid \tau \leq \rho' < \rho \rangle$ . Define  $A_{i^*}^{0\rho}$  to be the closure under the Skolem functions and  $\kappa^{+\rho}$ -sequences (or  $< \kappa^{+\rho}$ -sequences, if  $\rho^* = \rho + n$  for an inaccessible  $\rho^*$  and  $n < \omega$ ) of  $\tilde{A}_{i^*}^{0\rho}, \langle F_i^{\rho'} \mid \rho \leq \rho' \leq \delta, i < i^* \rangle$



and  $\langle F_i^{\rho'^*} \mid \rho \leq \rho' \leq \delta, i < i^* \rangle$ . Let  $\kappa(\rho) = \kappa^{+\rho}$  if there is an inaccessible  $\rho^* \leq \rho$  and  $n < \omega$  such that  $\rho = \rho^* + n$ , and let  $\kappa(\rho) = \kappa^{+\rho+1}$  otherwise. For a limit  $\alpha$ ,  $0 < \alpha < \kappa(\rho)$  let  $A_{i^*\alpha}^{0\rho} = \bigcup_{\alpha' < \alpha} A_{i^*\alpha'}^{0\rho}$ . Let  $A_{i^*\alpha+1}^{0\rho}$  be the closure of  $A_{i^*\alpha}^{0\rho} \cup \{A_{i^*\alpha}^{0\rho}\}$  under the Skolem functions and  $< \kappa(\rho)$ -sequences, for every  $\alpha < \kappa(\rho)$ . We set  $A_{i^*\alpha}^{0\rho} = \bigcup_{\alpha < \kappa(\rho)} A_{i^*\alpha}^{0\rho}$  and  $A_{i^*}^{1\rho} = \bigcup_{i < i^*} A_i^{1\rho} \cup \{A_{i^*\alpha}^{0\rho} \mid \alpha < \kappa(\rho)\} \cup \{A_{i^*}^{0\rho}\}$ .

**Case 2**  $i^*$  is a successor ordinal or  $i^*$  is a limit ordinal of cofinality  $\kappa^{+\tau+1}$  or less than  $\kappa^{+\tau}$  if  $\tau = \tau' + n$  for an inaccessible  $\tau'$  and  $n < \omega$ .

In this case we treat  $\tau$  in the same way as any other  $\rho \in (\tau, \delta]$  in the previous case. The definition for  $\rho \in (\tau, \delta]$  is as in Case 1.

Let us show now that such defined  $\langle \langle A_{i^*}^{0\rho}, A_{i^*}^{1\rho} \rangle \mid \tau \leq \rho \leq \delta \rangle$  is in  $\mathcal{P}'$ . Basically, we need to check the conditions (f) and (g) of Definition 1.1.

We start with (f). Let  $\tau \leq \rho \leq \rho' \leq \delta$ ,  $A \in A_{i^*}^{1\rho}$  and  $B \in A_{i'}^{1\rho'}$ . If  $A \in A_i^{1\rho}$  and  $B \in A_{i'}^{1\rho'}$  for some  $i, i' < i^*$ , then we use (f) for  $\langle \langle A_i^{0\nu}, A_i^{1\nu} \rangle \mid \tau \leq \nu \leq \delta \rangle$  where  $\bar{i} = \max(i, i')$ . It provides  $\rho \leq \tau'_1 \leq \dots \leq \tau'_\ell \leq \delta$ ,  $B_1 \in A \cap A_i^{1\tau'_1}, \dots, B_\ell \in A \cap A_i^{1\tau'_\ell}$  such that  $B \cap A = B_1 \cap \dots \cap B_\ell \cap A$ . Now, since  $A_i^{1\tau'_k} \subseteq A_{i^*}^{1\tau'_k}$  for every  $1 \leq k \leq \ell$  we are done.

If  $A \in A_i^{1\rho}$  for some  $i < i^*$  and  $B \in A_{i^*}^{1\rho'} \setminus \bigcup_{i < i^*} A_i^{1\rho'}$  then  $B \supseteq \bigcup_{i' < i^*} A_{i'}^{0\rho'}$ . In particular,  $B \supseteq A_{i^*}^{0\rho'} \supseteq A_{i^*}^{0\rho}$ . If  $A \in A_{i^*}^{1\rho} \setminus \bigcup_{i < i^*} A_i^{1\rho}$  and  $B \in A_{i'}^{1\rho'}$  for some  $i' < i^*$ , then we can use 1.1(g) for  $A, B$  and  $\langle \langle A_{i'}^{0\tau'}, A_{i'}^{1\tau'} \rangle \mid \tau' \leq \delta \rangle \in \mathcal{P}'$ . If  $A \in A_{i^*}^{1\rho} \setminus \bigcup_{i < i^*} A_i^{1\rho}$  and  $B \in A_{i^*}^{1\rho'} \setminus \bigcup_{i < i^*} A_i^{1\rho'}$ , then either  $B \supseteq A$  or  $B \subset A$  and in this case  $\rho' = \rho$  and  $B \in A$ .

Now let us check the condition (g). Thus let  $A$  be an elementary submodel of  $H(\kappa^{+\delta+2})$  of cardinality  $|A_{i^*}^{0\rho}|$ , closed under  $< |A_{i^*}^{0\rho}|$ -sequences,  $|A_{i^*}^{0\rho}| \in A$  and including  $\langle \langle A_{i^*}^{0\tau'}, A_{i^*}^{1\tau'} \rangle \mid \tau' \leq \delta \rangle$  as an element, for some  $\rho \leq \delta$ . Let  $\tau' \in [\rho, \delta]$  and  $B \in A_{i^*}^{1\tau'}$ . Suppose first that  $B \in A_{i'}^{1\tau'}$  for some  $i' < i^*$ . Then,  $\langle \langle A_{i'}^{0\nu}, A_{i'}^{1\nu} \rangle \mid \nu \leq \delta \rangle \in A$ , since  $A_{i^*}^{0\tau} \subseteq A_{i^*}^{0\rho} \subseteq A$  and the sequence  $\langle \langle A_{i'}^{0\nu}, A_{i'}^{1\nu} \rangle \mid \nu \leq \delta \rangle \in A_{i^*}^{0\tau}$ . So (g) of 1.1 applies to  $A, B$  and  $\langle \langle A_{i'}^{0\nu}, A_{i'}^{1\nu} \rangle \mid \nu \leq \delta \rangle$  and we are done. Assume now that  $B \in A_{i^*}^{1\tau'} \setminus \bigcup_{i < i^*} A_i^{1\tau'}$ . If  $\tau' = \rho$ , then  $B \in A$  since  $A \supseteq |A_{i^*}^{0\rho}| = \kappa(\rho)$ ,  $A_{i^*}^{0\rho} \in A$  and, so  $A_{i^*}^{0\rho} \subseteq A$ . But either  $B = A_{i^*}^{0\rho}$  or  $B \in A_{i^*}^{0\rho}$ . Suppose now that  $\tau' > \rho$ . If  $\tau' \in A$ , then  $A_{i^*}^{0\tau'} \in A$ . Recall that in this case  $A_{i^*}^{0\tau'} = \bigcup_{\alpha < \kappa(\tau')} A_{i^*\alpha}^{0\tau'}$  and  $A_{i^*}^{1\tau'} = \bigcup_{i < i^*} A_i^{1\tau'} \cup \{A_{i^*\alpha}^{0\tau'} \mid \alpha < \kappa(\tau')\} \cup \{A_{i^*}^{0\tau'}\}$ . If  $B = A_{i^*}^{0\tau'}$  or  $B = A_{i^*\alpha}^{0\tau'}$  for some  $\alpha \in A$  then we are done. Suppose otherwise. Then, let  $B = A_{i^*\alpha}^{0\tau'}$  for some  $\alpha < \kappa(\tau')$ . Set  $\tilde{\alpha} = \min(A \setminus \alpha)$ . Then  $\tilde{\alpha} \leq \kappa(\tau')$  and  $A_{i^*\tilde{\alpha}}^{0\tau'} \in A$ , where  $A_{i^*\kappa(\tau')}^{0\tau'} = A_{i^*}^{0\tau'}$ . But now  $A \cap B = A \cap A_{i^*\tilde{\alpha}}^{0\tau'}$ , since the chain  $\langle A_{i^*\beta}^{0\tau'} \mid \beta < \kappa(\tau') \rangle \in A$ . The rest of the proof follows completely those of [Gi, 4.18], where only the use of [Gi, 4.13] there is replaced by the following lemma:

**Lemma 1.10** *Let  $\tau \leq \delta$ . Suppose that  $\langle \langle A_i^{0\rho}, A_i^{1\rho} \rangle \mid \tau \leq \rho \leq \delta, i < \kappa(\tau)^+ \rangle$  (where  $\kappa(\tau) = \kappa^{+\tau'+n}$ , if  $\tau = \tau' + n$  for inaccessible  $\tau'$  and  $n < \omega$ , and  $\kappa(\tau) = \kappa^{+\tau+1}$  otherwise) is an increasing sequence of elements of  $\mathcal{P}'_{\geq \tau}$ , satisfying the following:*

For every  $i < \kappa(\tau)^+$  of cofinality  $\kappa(\tau)$

- (a)  $A_i^{0\tau} = \bigcup_{j < i} A_j^{0\tau}$ .
- (b) if  $B \in A_i^{1\rho}$  then either  $B \supseteq A_i^{0\tau}$  or  $B \in \bigcup_{j < i} A_j^{1\rho}$ .

Then for every  $i < \kappa(\tau)^+$  of cofinality  $\kappa(\tau)$  and  $B \in \bigcup\{A_j^{1\rho} \mid j \geq i, \tau \leq \rho \leq \delta\}$ , either  $B \supseteq A_i^{0\tau}$  or there are  $\tilde{i} < i$ ,  $\tau \leq \tau_1 \leq \tau_2 \leq \dots \leq \tau_\ell \leq \delta$  ( $\ell < \omega$ ) and  $B_1 \in A_{\tilde{i}}^{0\tau} \cap A_{\tilde{i}}^{1\tau_1}, \dots, B_\ell \in A_{\tilde{i}}^{0\tau} \cap A_{\tilde{i}}^{1\tau_\ell}$  such that for every  $j, \tilde{i} \leq j \leq i$

$$B \cap A_j^{0\tau} = B_1 \cap \dots \cap B_\ell \cap A_j^{0\tau}.$$

**Proof.** Fix  $i$  of cofinality  $\kappa(\tau)$ . Let  $B \in A_j^{1\rho}$  for some  $j, i \leq j < \kappa(\tau)^+$  and  $\rho$ , to  $A_i^{0\tau} \in A_j^{0\tau}$ ,  $\langle A_j^{0\nu}, A_j^{1\nu} \mid \tau \leq \nu \leq \delta \rangle$  and  $B \in A_j^{1\rho}$ . There are  $\ell < \omega$ ,  $\tau \leq \tau_1 \leq \dots \leq \tau_\ell \leq \delta$  and  $B_1 \in A_i^{0\tau} \cap A_j^{1\tau_1}, \dots, B_\ell \in A_i^{0\tau} \cap A_j^{1\tau_\ell}$  such that

$$B \cap A_i^{0\tau} = B_1 \cap \dots \cap B_\ell \cap A_i^{0\tau}.$$

By (a), there is  $\tilde{i} < i$  such that  $B_1, \dots, B_\ell \in A_{\tilde{i}}^{0\tau}$ . Fix  $k, 1 \leq k \leq \ell$ .  $B_k \in A_{\tilde{i}}^{0\tau} \subseteq A_{\tilde{i}}^{0\tau_k}$ , so  $B_k \subset A_{\tilde{i}}^{0\tau_k}$  since they have the same cardinality  $\kappa(\tau_k) \subseteq A_{\tilde{i}}^{0\tau_k}$ . Then by 1.2,  $B_k \in A_{\tilde{i}}^{1\tau_k}$ .

Clearly, for every  $i', \tilde{i} \leq i' \leq i$   $B \cap A_{i'}^{0\tau} = B \cap (A_i^{0\tau} \cap A_{i'}^{0\tau}) = (B \cap A_i^{0\tau}) \cap A_{i'}^{0\tau} = ((B_1 \cap \dots \cap B_k) \cap A_i^{0\tau}) \cap A_{i'}^{0\tau} = (B_1 \cap \dots \cap B_k) \cap (A_i^{0\tau} \cap A_{i'}^{0\tau}) = B_1 \cap \dots \cap B_k \cap A_{i'}^{0\tau}$ .

□

**Lemma 1.11** *Let  $\tau \leq \delta$ . Then the following holds:*

- (a) if there is no inaccessible  $\tau' < \tau$  and  $n < \omega$  such that  $\tau = \tau' + n$ , then  $\mathcal{P}_{<\tau}$  satisfies  $\kappa^{+\tau+2}$ -c.c. in  $V^{\mathcal{P}_{\geq\tau}}$
- (b) if  $\tau = \tau' + n$  for some inaccessible  $\tau' < \tau$  and  $0 < n < \omega$ , then  $\mathcal{P}_{<\tau}$  satisfies  $\kappa^{+\tau+1}$ -c.c. in  $V^{\mathcal{P}_{\geq\tau}}$ .
- (c) if  $\tau$  is an inaccessible, then  $\mathcal{P}_{<\tau}$  satisfies  $\kappa^{+\tau+2}$ -c.c. in  $V^{\mathcal{P}_{\geq\tau}}$ .

The proof of this lemma repeats the proof of 4.19 of [Gi]. We have here three cases because of different cardinalities according to the distance from an inaccessible.

**Lemma 1.12** *The forcing  $\mathcal{P}$  preserves all the cardinals except probably the successors of inaccessibles.*

This follows from 1.11 and 1.12.

**Remark 1.13** If one wants to preserve all the cardinals, then instead of the full support taken here, Easton type of support should be taken. Thus, fix some  $\langle \underline{A}^{0\nu}, \underline{A}^{1\nu}, \underline{F}^\nu \mid \nu \leq \delta \rangle \in \mathcal{P}$ . Let  $\underline{\mathcal{P}}$  consist of elements having Easton type support over the fixed condition, i.e.  $\langle \langle B^{0\nu}, B^{1\nu}, G^\nu \mid \nu \leq \delta \rangle$  will be in  $\underline{\mathcal{P}}$ , iff for every inaccessible  $\lambda \leq \delta$ ,  $|\{\nu \mid \langle B^{0\nu}, B^{1\nu}, G^\nu \rangle \neq \langle \underline{A}^{0\nu}, \underline{A}^{1\nu}, \underline{F}^\nu \rangle\}| < \lambda$ .

Now we define our main forcing  $\langle \mathcal{P}^{**}, \rightarrow \rangle$  as in [Gi]. Namely, let  $G \subseteq \mathcal{P}$  be generic. Set  $\mathcal{P}^{**} = \cup \{F^0 \mid \exists A^{00}, A^{10}, \langle \langle A^{0\tau}, A^{1\tau}, F^\tau \rangle \mid 0 < \tau \leq \delta \rangle \langle \langle A^{0\nu}, A^{1\nu}, F^\nu \rangle \mid \nu \leq \delta \rangle \in G\}$ .

The proof of the final lemma repeats those of [Gi].

**Lemma 1.14** *In  $V^{\mathcal{P}}$ ,  $\langle \mathcal{P}^{**}, \rightarrow \rangle$  satisfies  $\kappa^{++}$ -c.c.*

## 2 On gaps of size $\kappa^+$

The aim of the present section will be to sketch the proof of the following:

**Theorem 2.1** *Let  $\kappa$  be a cardinal of cofinality  $\omega$ . Suppose that for every  $\nu < \kappa$  the set  $\{\alpha < \kappa \mid o(\alpha) \geq \alpha^{+\nu}\}$  is cofinal in  $\kappa$ . Then there is a cofinality-preserving extension having the same bounded subsets of  $\kappa$  and satisfying  $2^\kappa = \kappa^{+\delta+1}$  for every  $\delta < \kappa^{++}$ .*

For  $\delta < \kappa$  it was done in [Gi], so we concentrate on  $\delta$ 's between  $\kappa$  and  $\kappa^{++}$ .

Pick an increasing sequence  $\kappa_0 < \kappa_1 < \dots < \kappa_n < \dots < \kappa$  so that

- (a)  $\bigcup_{n < \omega} \kappa_n = \kappa$
- (b)  $\kappa_n$  carries an extender of the length  $\kappa_n^{+\kappa_{n-1}}$ , for every  $n$ ,  $0 < n < \omega$ .

We force as in [Gi] but with the following correspondence function  $a_n$ :

- (i)  $\text{dom} a_n \subseteq \kappa^{+\kappa_{n-1}}$  and
- (ii)  $a_n(\kappa^{+\alpha+1}) = \kappa_n^{+\alpha+1}$  for  $\alpha \in [\kappa^+, \kappa^{+\kappa_{n-1}})$ .

This forcing produces an increasing (mod finite) sequence of functions  $\langle f_\alpha \mid \alpha \in [\kappa^+, \kappa^{+\kappa}] \rangle$  such that  $f_\alpha$  corresponds to  $\kappa^{+\alpha+1}$ . Just define  $f_\alpha(n)$  to be the element of the one element Prikry sequence corresponding to  $\kappa_n^{+\alpha+1}$ , for every  $n \geq \min\{m \mid \alpha \in [\kappa^{+m}, \kappa^{+\kappa}]\}$ . The generic extension will satisfy  $2^\kappa \geq \kappa^{+\kappa}$ .

In order to make  $2^\kappa \geq \kappa^{+\kappa^+}$  let us pick  $\kappa_n$ 's carrying extenders of length  $\kappa_n^{+\kappa_{n-1}^+}$ . We require here only that  $a_n(\kappa^{+\alpha}) = \kappa_n^{+a_{n-1}(\alpha)}$  and  $a_{n-1}(\alpha) \leq \kappa_{n-1}^+$  for every  $\alpha \leq \kappa^+$ . There is one minor problem that for some  $\alpha < \kappa^+$   $\kappa^{+\alpha+1}$  can be in  $\text{dom} f_n$ , where the conditions at level  $n$  are of the form  $\langle a_n, A_n, f_n \rangle$ . We required in [Gi] that  $\text{dom} a_n \cap \text{dom} f_n = \emptyset$ . Here we

need to deal with ordinals of cardinality  $\kappa^{+\alpha}$  and to add some of them to  $\text{dom}a_n$ . We can either do it directly and require  $|a_n(\beta_1)| = |a_n(\beta_2)|$  for any two such ordinals, or explicitly add  $\kappa^{+\alpha+1}$  to  $\text{dom}a_n$  and remove this value after a nondirect extension is taken. This produces a generic extension satisfying  $2^\kappa \geq \kappa^{+\kappa^+}$ .

Now in order to deal with arbitrary  $\delta \in [\kappa^+, \kappa^{++})$  we deal first with  $(\kappa^+)^m$  for every  $m < \omega$  and then use the Rado-Milnor paradox, see K. Kunen [Ku, Ch. 1, Ex. 20].

Fix  $m < \omega$ . We use  $\kappa_n$  which carries an extender of length  $\kappa_n^{+(\kappa_{n-1}^+)^m}$ . We proceed as above with the following addition:

for every  $\kappa^+ \leq \beta < (\kappa^+)^m$  let  $\beta = (\kappa^+)^{k_1} \cdot \alpha_1 + (\kappa^+)^{k_2} \cdot \alpha_2 + \dots + (\kappa^+)^{k_\ell} \cdot \alpha_\ell$  where  $\omega > k_1 > k_2 > \dots > k_\ell$ ,  $\ell < \omega$ ,  $\alpha_1, \dots, \alpha_\ell < \kappa^+$ , then we require that  $a_n(\beta) = (\kappa_{n-1}^+)^{k_1} \cdot a_n(\alpha_1) + (\kappa_{n-1}^+)^{k_2} \cdot a_n(\alpha_2) + \dots + (\kappa_{n-1}^+)^{k_\ell} \cdot a_n(\alpha_\ell)$ .

### 3 Doing below the first inaccessible above $\kappa$

A straightforward application of the techniques developed in [Gi] and here, can get one above  $\kappa^{+\kappa^{+\nu+2}}$  for  $\nu < \kappa$  starting with  $o(\kappa_n) = \kappa_n^{+\kappa_n^{+n+\nu+2}} + 1$  ( $n < \omega$ ), above  $\kappa^{+\kappa^{+\kappa^{+\nu+2}}}$  starting with  $o(\kappa_n) = \kappa_n^{+\kappa_n^{+\kappa_n^{+n+\nu+2}}} + 1$ , etc.

Let us explain this dealing with  $\kappa^{+\kappa^{+\nu+2}}$ , i.e. we want to have  $2^\kappa \geq \kappa^{+\kappa^{+\nu+2}}$  using an extender of length  $\kappa_n^{+\kappa_n^{+n+\nu+2}}$  for each  $n < \omega$ . The definition of the preparation forcing  $\mathcal{P}$  is as 1.5 with the change in the cardinals correspondence condition. We require the following:

- (i)  $a_n(\kappa^{+\tau+1}) = \kappa_n^{+n+\tau+1}$
- (ii)  $a_n(\kappa^{+\kappa^{+\tau+1}}) = \kappa_n^{+a_n(\kappa^{+\tau+1})}$   
for every  $\tau \leq \nu$  and  $a_n$  as in 1.5.

The rest of the construction is the same. Notice only that in a previous section at each  $\kappa_n$  we had an extender of inaccessible length which allowed us to have many similar cardinals in the interval  $(\kappa_n, \text{the first inaccessible above } \kappa_n)$ . This in turn allowed us to pick the correspondence between cardinals in the intervals  $(\kappa, \kappa^{+\delta})$  and in  $(\kappa_n, \text{the first inaccessible above } \kappa_n)$  generically. In the present situation the number of cardinals to deal with is relatively small and so we use (i) and (ii) above to define the correspondence. It remains [Gi, sec. 4,5], where the correspondence was also defined in advance.

Now we like to implement the Shelah idea [Sh1] in order to show that below the first inaccessible above  $\kappa$  any gap is possible, provided that for every  $n < \omega$  we have an extender over  $\kappa_n$  of the strength of the fixed point of the aleph function above  $\kappa_n$  of order  $n$ . Let us first recall the definition.

**Definition 3.1** (Shelah [Sh1]). Let  $C^0 =$  the class of all infinite cardinals.  $C^{n+1} = \{\lambda \in C^n \mid C^n \cap \lambda \text{ has order type } \lambda\}$  and  $C^\omega = \bigcap_{n < \omega} C^n$ . The order of a cardinal  $\nu$  is the maximal  $n \leq \omega$  such that  $\nu \in C^n$ . Elements of  $C^n (n \geq 1)$  are called fixed points of the  $\aleph$ -function of order  $n$ .

**Theorem 3.2** *Suppose that  $\kappa = \bigcup_{n < \omega} \kappa_n$ ,  $\kappa_0 < \kappa_1 < \dots < \kappa_n < \dots$ ,  $o(\kappa_n) =$  (the first fixed point of the  $\aleph$ -function above  $\kappa_n$  of order  $n$ )+1 and there is no inaccessible cardinal above  $\kappa$ . Then for every  $\lambda$  there are cardinals and cofinalities preserving the extension, not adding new bounded subsets to  $\kappa$  and satisfying  $2^\kappa \geq \lambda$ .*

**Proof.** Let  $\mu > \kappa$ . By S. Shelah [Sh1], Lemma 2.5 there exists an increasing sequence  $\langle D_n \mid n < \omega \rangle$  so that  $\bigcup_{n < \omega} D_n = \{\chi \mid \chi \text{ is a cardinal } \kappa^{++} \leq \chi \leq \mu^+\}$  and for every  $n < \omega$  there is no elements of  $C^n$  between  $\kappa$  and  $\mu^+$  in a generic extension  $V_n$  of  $V$  obtained by preserving only elements of  $D_n$  as cardinals between  $\kappa^{++}$  and  $\mu^+$ . Without loss of generality, let us assume that  $\kappa^{++}$  and  $\mu^+$  are in  $D_0$ .

We like to correspond elements of  $D_n$  to the cardinals between  $\kappa_n^{+n+2}$  and the length of extender over  $\kappa_n$ , i.e. the least fixed point above  $\kappa_n$  of the  $\aleph$ -function of order  $n$ . Once we have this correspondence, the rest of the construction will be as in the previous section.

Let us define by induction on  $n < \omega$  a two place function  $f_n$  from ordinals into cardinals. For every ordinal  $\xi$  set  $f_0(\xi, 0) = \xi$ ,  $f_0(\xi, 1) = \xi^+$ ,  $f_0(\xi, \alpha + 1) = (f_0(\xi, \alpha))^+$  and  $f_0(\xi, \alpha) = \bigcup_{\beta < \alpha} f_0(\xi, \beta)$  for a limit  $\alpha$ . I.e.  $f_0(\xi, \alpha) = \xi^{+\alpha}$ , the  $\alpha$ -th cardinal past  $\xi$ . Now define  $f_1$ . First, for an ordinal  $\xi$  we define  $f_1(\xi, 0)$ . Set  $\rho_0 = \xi$ ,  $\rho_1 = f_0(\xi, \rho_0)$  and  $\rho_{n+1} = f_0(\xi, \rho_n)$  for every  $n < \omega$ . Let  $f_1(\xi, 0) = \bigcup_{n < \omega} \rho_n$ . Then,  $f_1(\xi, 0)$  will be the least fixed point of the  $\aleph$ -function above  $\xi$ . We define  $f_1(\xi, 1)$  in a similar fashion to be the second fixed point of the  $\aleph$ -function above  $\xi$ . Thus, set  $\rho_0 = f_1(\xi, 0)$ ,  $\rho_1 = f_0(f_1(\xi, 0), \rho_0)$  and  $\rho_{n+1} = f_0(f_1(\xi, 0), \rho_n)$  for every  $n < \omega$ . Set  $f_1(\xi, 1) = \bigcup_{n < \omega} \rho_n$ . Clearly,  $f_1(\xi, 1) = f_1(f_1(\xi, 0), 0)$ . Now for every limit  $\alpha$  let  $f_1(\xi, \alpha) = \bigcup_{\beta < \alpha} f_1(\xi, \beta)$ . Define  $f_1(\xi, \alpha + 1)$  to be least fixed point of the  $\aleph$ -function above  $f_1(\xi, \alpha)$ , i.e.  $f_1(\xi, \alpha + 1) = f_1(f_1(\xi, \alpha), 0)$ .

Suppose now that for every  $k \leq m$   $f_k$  is defined and  $f_k(\xi, \alpha)$  is the  $\alpha$ -th fixed point of the order  $k$  above  $\xi$ . Define  $f_{m+1}$ . First for any ordinal  $\xi$  we define  $f_{m+1}(\xi, 0)$ . Set  $\rho_0 = f_m(\xi, 0)$ ,  $\rho_1 = f_m(f_m(\xi, 0), \rho_0)$  and  $\rho_{n+1} = f_m(f_m(\xi, 0), \rho_n)$  for every  $n < \omega$ . Let  $f_{m+1}(\xi, 0) = \bigcup_{n < \omega} \rho_n$ . For a limit ordinal  $\alpha$  we set  $f_{m+1}(\xi, \alpha) = \bigcup_{\beta < \alpha} f_m(\xi, \beta)$ . At successor stage let  $f_{m+1}(\xi, \alpha + 1) = f_{m+1}(f_{m+1}(\xi, \alpha), 0)$ . This completes the inductive definition of  $\langle f_n \mid n < \omega \rangle$ .

Now we are going to use such defined functions  $\langle f_n \mid n < \omega \rangle$  in order to produce the desired correspondence between elements of  $D_n$ 's and cardinals between  $\kappa_n^{+n+2}$  and the fixed

point of order  $n$  of the  $\aleph$ -function above  $\kappa_n$ .

Fix  $n < \omega$ . We may either work in the world obtained by leaving only elements of  $D_n$  to be the cardinals in the interval  $(\kappa^{++}, \mu^+)$  or relating the functions to  $f_m(m < \omega)$  to the set  $D_n$  in the obvious fashion. Let us deal with the first possibility. Notation in this case will be a bit simpler. Thus, we assume that the cardinals between  $\kappa^{++}$  and  $\mu^+$  are only the elements of  $D_n$ . Then, by the choice of  $D_n$ , there is no fixed points of the  $\aleph$ -function of order  $n$  between  $\kappa^{++}$  and  $\mu^+$ . We define inductively the correspondence  $\pi$ . For every  $\alpha \leq \kappa_n$  let  $\pi(\alpha) = \alpha$ . For  $\alpha$ ,  $\kappa_n < \alpha < \kappa^+$   $\pi(\alpha)$  may take any value in the interval  $(\kappa_n, \kappa_n^{+n+1})$ .  $\pi(\kappa^+) = \kappa_n^{+n+1}$  and  $\pi(\kappa^{++}) = \kappa_n^{+n+2}$ . Notice that  $\kappa^{++} = f_0(\kappa^{++}, 0)$ . So, we have defined the correspondence between sets  $\{0, 1, \dots, \alpha, \dots, \kappa, \kappa^+, \kappa^{++} = f_0(\kappa^{++}, 0)\}$  and  $\{0, 1, \dots, \alpha, \dots, \kappa_n, \kappa_n^+, \dots, \kappa_n^{+n}, \kappa_n^{+n+1}, \kappa_n^{+n+2} = f_0(\kappa_n^{+n+2}, 0)\}$ . Using it we continue to the sets  $\{f_0(\kappa^{++}, 0), f_0(\kappa^{++}, 1), \dots, f_0(\kappa^{++}, \alpha), \dots, f_0(\kappa^{++}, f_0(\kappa^{++}, 0))\}$  and  $\{f_0(\kappa_n^{+n+2}, 0), f_0(\kappa_n^{+n+2}, 1), \dots, f_0(\kappa_n^{+n+2}, \alpha), \dots, f_0(\kappa_n^{+n+2}, f_0(\kappa_n^{+n+2}, 0))\}$ . Just  $f_0(\kappa^{++}, \alpha)$  will correspond to  $f_0(\kappa_n^{+n+2}, \pi(\alpha))$ . Set  $\rho_1 = f_0(\kappa^{++}, f_0(\kappa^{++}, 0))$  and  $\bar{\rho}_1 = f_0(\kappa_n^{+n+2}, f_0(\kappa_n^{+n+2}, 0))$ . Then  $\pi(\rho_1) = \bar{\rho}_1$ . We now consider the sets  $\{f_0(\rho_1, 0), f_0(\rho_1, 1), \dots, f_0(\rho_1, \alpha), \dots, f_0(\rho_1, \rho_1)\}$  and  $\{f_0(\bar{\rho}_1, 0), f_0(\bar{\rho}_1, 1), \dots, f_0(\bar{\rho}_1, \alpha), \dots, f_0(\bar{\rho}_1, \bar{\rho}_1)\}$ . Extend  $\pi$  to these sets by setting  $\pi(f_0(\rho_1, \alpha)) = f_0(\bar{\rho}_1, \pi(\alpha))$ . Let  $\rho_2 = f_0(\rho_1, \rho_1)$  and  $\pi(\rho_2) = \bar{\rho}_2 = f_0(\bar{\rho}_1, \bar{\rho}_1)$ . We consider the sets  $\{f_0(\rho_2, 0), \dots, f_0(\rho_2, \rho_2)\}$  and  $\{f_0(\bar{\rho}_2, 0), \dots, f_0(\bar{\rho}_2, \bar{\rho}_2)\}$ . Deal with them in the same fashion. Continuing and using induction, he will be able to extend  $\pi$  up to  $\rho_\omega = \bigcup_{n < \omega} \rho_n$ . But, clearly,  $\rho_\omega = f_1(\kappa^{++}, 0)$ , i.e. the first fixed point of the  $\aleph$ -function above  $\kappa^{++}$ . So we have the correspondence between sets  $\{0, \dots, \kappa, \dots, f_1(\kappa^{++}, 0)\}$  and  $\{0, \dots, \kappa_n, \dots, f_1(\kappa_n^{+n+2}, 0)\}$ . Let us extend it to the correspondence between the sets  $\{f_1(\kappa^{++}, 0), f_1(\kappa^{++}, 1), \dots, f_1(\kappa^{++}, \alpha), \dots, f_1(\kappa^{++}, f_1(\kappa^{++}, 0))\}$  and  $\{f_1(\kappa_n^{+n+2}, 0), f_1(\kappa_n^{+n+2}, 1), \dots, f_1(\kappa_n^{+n+2}, \alpha), \dots, f_1(\kappa_n^{+n+2}, f_1(\kappa_n^{+n+2}, 0))\}$ . Just let  $f_1(\kappa^{++}, \alpha)$  correspond to  $f_1(\kappa_n^{+n+2}, \pi(\alpha))$ . The correspondence between the intervals  $(f_1(\kappa^{++}, \alpha), f_1(\kappa^{++}, \alpha + 1))$  and  $(f_1(\kappa_n^{+n+2}, \pi(\alpha)), f_1(\kappa_n^{+n+2}, \pi(\alpha) + 1))$  is defined using  $f_0(f_1(\kappa^{++}, \alpha), \dots)$ , as above. Set  $\rho_1 = f_1(\kappa^{++}, f_1(\kappa^{++}, 0))$  and  $\bar{\rho}_1 = \pi(\rho_1) = f_1(\kappa_n^{+n+2}, f_1(\kappa_n^{+n+2}, 0))$ . We consider the sets  $\{f_1(\rho_1, 0), f_1(\rho_1, 1), \dots, f_1(\rho_1, \alpha), \dots, f_1(\rho_1, f_1(\rho_1, 0))\}$  and  $\{f_1(\bar{\rho}_1, 0), f_1(\bar{\rho}_1, 1), \dots, f_1(\bar{\rho}_1, \alpha), \dots, f_1(\bar{\rho}_1, f_1(\bar{\rho}_1, 0))\}$ . Extend  $\pi$  to these sets as above. Let  $\rho_2 = f_1(\rho_1, f_1(\rho_1, 0))$  and  $\bar{\rho}_2 = \pi(\rho_2) = f_1(\bar{\rho}_1, f_1(\bar{\rho}_1, 0))$ . Continue in the same fashion by induction we will obtain  $\rho_{m+1} = f_1(\rho_m, f_1(\rho_m, 0))$ ,  $\bar{\rho}_{m+1} = \pi(\rho_{m+1}) = f_1(\bar{\rho}_m, f_1(\bar{\rho}_m, 0))$  and the extension of  $\pi$  to the intervals  $[0, \rho_{m+1}]$ ,  $[0, \bar{\rho}_{m+1}]$ , where  $m < \omega$ . Set  $\rho_\omega = \bigcup_{m < \omega} \rho_m$ . Then  $\rho_\omega = f_2(\kappa^{++}, 0)$  will be the first fixed point of the  $\aleph$ -function of the order 2 above  $\kappa^{++}$ . By induction it is easy to continue the process up to  $f_n(\kappa^{++}, 0)$  (the first fixed point of the  $\aleph$ -function of the order  $n$  above  $\kappa^{++}$ ).

This completes the inductive definition of correspondence  $\pi$ .

The rest of the construction repeats those of Section 1.

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